Retrospectives on My Studies of Solid Mechanics (I)
- variational basis of solid mechanics -

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ABSTRACT

The present author has proposed unified principle of virtual work and complementary virtual work quite recently [(1),(2)].
It was found using the divergence theorem in elasticity that twice of the strain energy to be stored in an elastic body is equal to sum of work done (potential) due to not only a given body force, surface traction acting on the stress boundary but also the enforced displacement on the displacement boundary.

Then using this obvious relation, a new stationary energy principle can be proposed which unifies the principles of virtual work and complementary virtual work. In case of the linear elasticity problems, it becomes the minimum principle of total energy of a given elastic system which is sum of potential and complementary energy of the said system. It can be also shown that the lower bound solution of the same system can be always obtained using this new principle with the displacement function which satisfies the equation of equilibrium.

1. Introduction

According to historical survey [(8),(9)], the principle of minimum potential energy was formulated by J. Willard Gibbs in 1875 and the complementary energy concept was introduced by F. Z. Engesser in 1889. It is well known that the former can give the upper bound solution, while the later can give the lower bound solution of the true solutions in elasticity problems. It is very strange that both principles were independently proposed and because of easier usage, the former has been well established by the middle of last century.
Jon Turners' paper on the Direct Stiffness Method published in 1956 has become the origin of the present finite element method where the element displacement functions are assumed unknown [(5),(6)] .
In early stage of the finite element displacement, around 1950's, however, the force method (equilibrium method) existed together where element forces are assumed unknowns.
However due to rapid development of the Displacement Method, the Force Method declined quickly and today the displacement method represents almost all of the finite element method.

Superscript () implies the number of literature quoted in the reference.

In 1965 B. M. Fraeijs de Veubeke [(4)] discussed the variational basis of the finite element method from both DM and FM standpoint of view, and he developed a general variational principle combining the potential energy and the dislocation potential which he gave the name.
Almost at the same time T. H. H. Pian [(3),(5)] developed the hybrid or mixed variational methods to establish a unified basis of the finite element method.
Both methods, however, are based on Hellinger-Reissner's variational principle and therefore they can only give the stationary solution. And the mathematical basis (convergency studies and error estimate) of the finite element method is considered well established. It is, however, the Displacement Method which can only give the upper bound of true solutions.
Consequently it is obvious that restoration of the Force Method is imperative because it can give always the lower bound of the true solutions.

This is motivation of my research by which accuracy of approximate solution can be correctly estimated. For this purpose the author challenged on the development of the unified energy principle without using Lagrange multiplier.

2. Development of the unified principle of virtual work and complementary virtual work

Consider arbitrary sets of stress components σ_{ij} and strain components ε_{ij} of any solid subjected to external loading and enforced displacement. σ_{ij} are assumed to satisfy the following equation of equilibrium and the stress boundary condition:

$$\sigma_{ij,j} + \overline{p}_i = 0 \quad \text{in} \quad V \tag{1}$$

where $\overline{p}_i$ is a given body force vector and V is the volume of a given body and

$$t_i = \sigma_{ij} n_j \quad \text{on} \quad S_\sigma \tag{2}$$

where n_i is the unit normal drawn outward on the stress boundary S_σ, t_i is the traction vector on S_σ. (See Fig.1)
The strain ε_{ij} is assumed to be derived from the displacement using eq (3), and the displacement u_i is also assumed to satisfy the displacement boundary condition (4) as follows:

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \tag{3}$$

$$u_i = \overline{u}_i \quad \text{on} \quad S_u \tag{4}$$

where $S = S_\sigma + S_u$, S_u is the displacement boundary.
Using eq (3) and applying the well-known divergence theorem it is not too difficult to derive

the following equation:

$$\int_V \sigma_{ij}\varepsilon_{ij}dV = \int_S t_i u_i dS - \int_V \sigma_{ij,j}u_i dV \tag{5}$$

Applying eqs (1), (2) and (4) to eq(5), the following equation can be derived.

$$\int_V \sigma_{ij}\varepsilon_{ij}dV = \int_V \overline{p}_i u_i dV + \int_{S_\sigma} \bar{t}_i u_i dS + \int_{S_u} \overline{u}_i t_i dS \tag{6}$$

It should be mentioned here that this equation is true irrespective of the stress-strain relation and size of u_i and σ_{ij} and from which the following equation can be derived:

$$\delta\int_V \sigma_{ij}\varepsilon_{ij}dV = \int_V \sigma_{ij}\delta\varepsilon_{ij}dV + \int_V \varepsilon_{ij}\delta\sigma_{ij}dV = \int_V \overline{p}_i \delta u_i dV + \int_{S_\sigma} \bar{t}_i \delta u_i dS + \int_{S_u} \overline{u}_i \delta t_i dS$$

That is, in weak form:

$$\delta\int_V \sigma_{ij}\varepsilon_{ij}dV - \int_V \overline{p}_i \delta u_i dV - \int_{S_\sigma} \bar{t}_i \delta u_i dS - \int_{S_u} \overline{u}_i \delta t_i dS = 0 \tag{7-a}$$

or in strong form

$$\left(\int_V \sigma_{ij}\delta\varepsilon_{ij}dV - \int_V \overline{p}_i \delta u_i dV - \int_{S_\sigma} \bar{t}_i \delta u_i dS\right) + \left(\int_V \varepsilon_{ij}\delta\sigma_{ij}dV - \int_{S_u} \overline{u}_i \delta t_i dS\right) = 0 \tag{7-b}$$

Eq (7-a,b) may be unified principle of the virtual work and complementary virtual work. (Fig. 2) Namely, if u_i and σ_{ij} are true solutions, the following two variational equations can be realized:

(a) the principle of virtual work

$$\int_V \sigma_{ij}\delta\varepsilon_{ij}dV - \int_V \overline{p}_i \delta u_i dV - \int_{S_\sigma} \bar{t}_i \delta u_i dS = 0 \quad (\text{w.r.t. } u_i) \tag{8}$$

(b) the principle of complementary virtual work

$$\int_V \varepsilon_{ij}\delta\sigma_{ij}dV - \int_{S_u} \overline{u}_i \delta t_i dS = 0 \quad (\text{w.r.t. } \sigma_{ij}) \tag{9}$$

Therefore eq (7-a) is proved.

Conversely if eq (7-a) is true eq (7-b) is also true with respect to u_i and σ_{ij}.

Since u_i and σ_{ij} are assumed independently eqs (8) and (9) must be realized simultaneously.

Process for proposing the unified principle of virtual work and complementary virtual work may be illustrated in the following Fig. 2.

3. Proposition of the unified principle of total energy in the linear elasticity

If σ_{ij} and ε_{ij} are related by the following linear relation:

$$\sigma_{ij} = a_{ijkl}\varepsilon_{kl} \quad \text{or} \quad \varepsilon_{ij} = b_{ijkl}\sigma_{kl} \tag{10}$$

where a_{ijkl} and b_{ijkl} are symmetric matrices, eq (6) expresses the law of energy conservation.

Now consider the following total energy of an elastic system as defined by

$$\Pi_t(u_i) = U - W \tag{11}$$

$$\text{where} \quad U = \int_V \sigma_{ij}\varepsilon_{ij}dV = V + V_C = 2V = 2V_C \tag{12-a}$$

$$\left.\begin{aligned} V &= \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}dV = \frac{1}{2}\int_V a_{ijkl}\varepsilon_{ij}\varepsilon_{kl}dV \\ V_C &= \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij}dV = \frac{1}{2}\int_V b_{ijkl}\sigma_{ij}\sigma_{kl}dV \end{aligned}\right\} \tag{12-b}$$

$$\text{and} \quad W = W_p + W_c \tag{13-a}$$

$$W_p = \int_V \overline{p}_i u_i dV + \int_{S_\sigma} \bar{t}_i u_i dS \tag{13-b}$$

$$W_c = \int_{S_u} \overline{u}_i t_i dS \tag{13-c}$$

Then eq (11) can be written as follows:

$$\Pi_t(u_i) = \Pi_p(u_i) + \Pi_c(u_i) \tag{14-a}$$

$$\text{where} \quad \Pi_p(u_i) = V_p(u_i) - W_p(u_i) = \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij}dV - \int_V \overline{p}_i u_i dV - \int_{S_\sigma} \bar{t}_i u_i dS \tag{14-b}$$

$$\Pi_c(\sigma_{ij}) = W_c(\sigma_{ij}) - V_c(\sigma_{ij}) = \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij}dV - \int_{S_u} \overline{u}_i t_i dS \tag{14-c}$$

It should be mentioned here that the complementary energy Π_c is originally defined with respect to σ_{ij}, but now it is a function of u_i because σ_{ij} is a linear function of u_i via eqs (3) and (10).

Therefore it can be concluded that if u_i is the true solution, by the minimum principle of potential and complementary energy, the following conclusion can be drawn:

$$\Pi_t(u_i) \rightarrow \min \quad \text{w.r.t. } u_i \tag{15}$$

Conversely consider the case where $\Pi_t(u_i)$ becomes minimum with respect to u_i.

Since $\Pi_t(u_i)$ is sum of two positive functional $\Pi_p(u_i)$ and $\Pi_c(u_i)$, if at least any one of

theme does not become minimum, then $\Pi_t(u_i)$ can not become minimum, Q.E.D.
Thus it can be concluded that the new principle proposed in this section unifies the minimum principles of potential and complementary energies.
Next, let's consider the strong form of $\delta\Pi_t(u_i)=0$.
Now it is given by

$$\int_V \left(\sigma_{ij}\delta\varepsilon_{ij} + \delta\sigma_{ij}\varepsilon_{ij}\right)dV - \int_V \bar{p}_i\delta u_i dV - \int_{S_\sigma} \bar{t}_i\delta u_i dS - \int_{S_u} \bar{u}_i\delta t_i dS = 0$$

This equation is further transformed using the divergence theorem as follows:

$$\int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS + \int_{S_u} (u_i - \bar{u}_i)\delta t_i dS - \int_V \left(\sigma_{ij,j} + \bar{p}_i\right)\delta u_i dV - \int_V \delta\sigma_{ij,j} u_i dV = 0 \qquad (17)$$

The last volume integral is physically interpreted as the complementary virtual work of $\delta\sigma_{ij,j}$ (virtual body force due to some physical actions such as heat conduction, fluid flow, electromagnetism and so on).
And therefore in case of pure mechanics problem, it may by deleted.

$$\therefore \quad \int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS + \int_{S_u} (u_i - \bar{u}_i)\delta t_i dS - \int_V \left(\sigma_{ij,j} + \bar{p}_i\right)\delta u_i dV = 0 \qquad (18)$$

4. Correlation study of a new variational principle derived and other existing principles.

It should be mentioned here that the author derived previously the following variational equation generalizing the principle of virtual work with the use of Lagrange multiplier:

$$\int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS - \int_{S_u} (u_i - \bar{u}_i)\delta t_i dS - \int_V \left(\sigma_{ij,j} + \bar{p}_i\right)\delta u_i dV = 0 \qquad (19)$$

This equation is the strong form of the following modified Hellinger-Reissner's variational equation [(4),(8)] :
$\delta\Pi_R\left(\sigma_{ij}, u_i, \lambda_i\right) = 0$ w.r.t. σ_{ij}, u_i and Lagrange multipliers λ_i

$$\text{where} \quad \Pi_R\left(\sigma_{ij}, u_i, \lambda_i\right) = \int_V \left(\sigma_{ij}\varepsilon_{ij} - \frac{1}{2}b_{ijkl}\sigma_{ij}\sigma_{kl} - \bar{p}_i u_i\right)dV - \int_{S_\sigma} \bar{t}_i u_i dS - \int_{S_u} \lambda_i\left(u_i - \bar{u}_i\right)dS \qquad (20)$$

That is, eq (20) is reduced to eq (19) if σ_{ij} and λ_i are related with u_i by eqs (2) and $\lambda_i = t_i$.
It is surprising to note that difference of eqs (18) and (19) is only sign of the second term of both equations, but eq (18) can give always the lower bound solution, while eq (19) can give only stationary solutions.
It should be also emphasized here that eq (18) is derived without introducing Lagrange

multiplier, and therefore it can guarantee the minimum property of $\Pi_t(u_i)$ while $\delta\Pi_R(\sigma_{ij}, u_i) = 0$, the strong form of which is given by eq (19) is only a stationary principle.

The followings are conclusion of this section:

(i) $\delta\Pi_p(u_i) = 0$ gives the upper bound solution in linear elasticity problems.

(ii) $$\Pi_t(u_i) = \Pi_p(u_i) - \int_{S_u} (u_i - \bar{u}_i) t_i ds \qquad (21)$$

$\delta\Pi_t(u_i) = 0$ can give the lower bound solutions.

(iii) $$\Pi_{HR}(u_i) = \Pi_p(u_i) + \int_{S_u} (u_i - \bar{u}_i) t_i ds \qquad (22)$$

$\delta\Pi_{HR}(u_i) = 0$ gives only the stationary solutions.

5. Eight possible methods of solution on the elasticity problems

In general the true solution of the boundary value problem of elasticity must satisfy the following three conditions:

$$\left.\begin{array}{lll}\text{(a) equilibrium condition:} & \sigma_{ij,j} + \bar{p}_i = 0 & \text{in } V \\ \text{(b) displacement boundary conditions:} & u_i = \bar{u}_i & \text{on } S_u \\ \text{(c) stress boundary conditions:} & t_i = \bar{t}_i & \text{on } S_\sigma\end{array}\right\} \qquad (23)$$

Considering possible combination of above three conditions, 8 different variational equations can be proposed for the approximate solution as shown in Fig 3 and Table 1.

For instance, in case of Rayleigh-Ritz method which is the second method of solution in Table 1, the displacement functions u_i for a given entire field is usually assumed in the form of truncated polynomials of coordinate variables x_i and the displacement boundary condition must be satisfied a priori. Using such a displacement function the total potential energy $\Pi_p(u_i)$ is computed, and it is generally given by a quadratic function of the unknown constants a_k of the assumed displacement function.

Then final linear equation of a_k to be solved can be obtained by computing $\dfrac{\partial\Pi}{\partial a_k} = 0$.

It should be mentioned here that the first and fifth methods of solution do not require both displacement and stress boundary conditions a priori. This makes the analysis much easier to compare the other 6 methods.

6. Conclusions

Using divergence theorem in elasticity, a new variational principle is proposed on the minimum condition of the total energy of a given system.

In this paper, the minimum principles of potential energy and complementary energy are unified without using Lagrange multiplier and therefore the minimum condition of the total energy is guaranteed.

Consequently, the lower bound solutions can be always obtained. In the subsequent series of articles, the lower bound solutions will be obtained on a set of different elasticity problems using several methods among 8 classified methods of solution.

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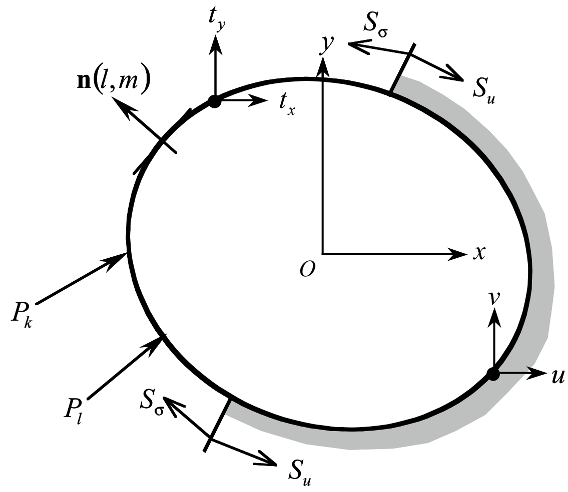


equilibrium equation:

$\sigma_{ij,j} + \bar{p}_i = 0$ in V

displacement b.c.: $u_i = \bar{u}_i$ on S_u

stress b.c.: $t_i = \bar{t}_i$ on S_σ

where $t_i = \sigma_{ij} n_j$

S_u : displacement prescribed condition

S_σ :stress prescribed condition

$S = S_u + S_\sigma$

Fig. 1 Boundary value problem of 2D elasticity

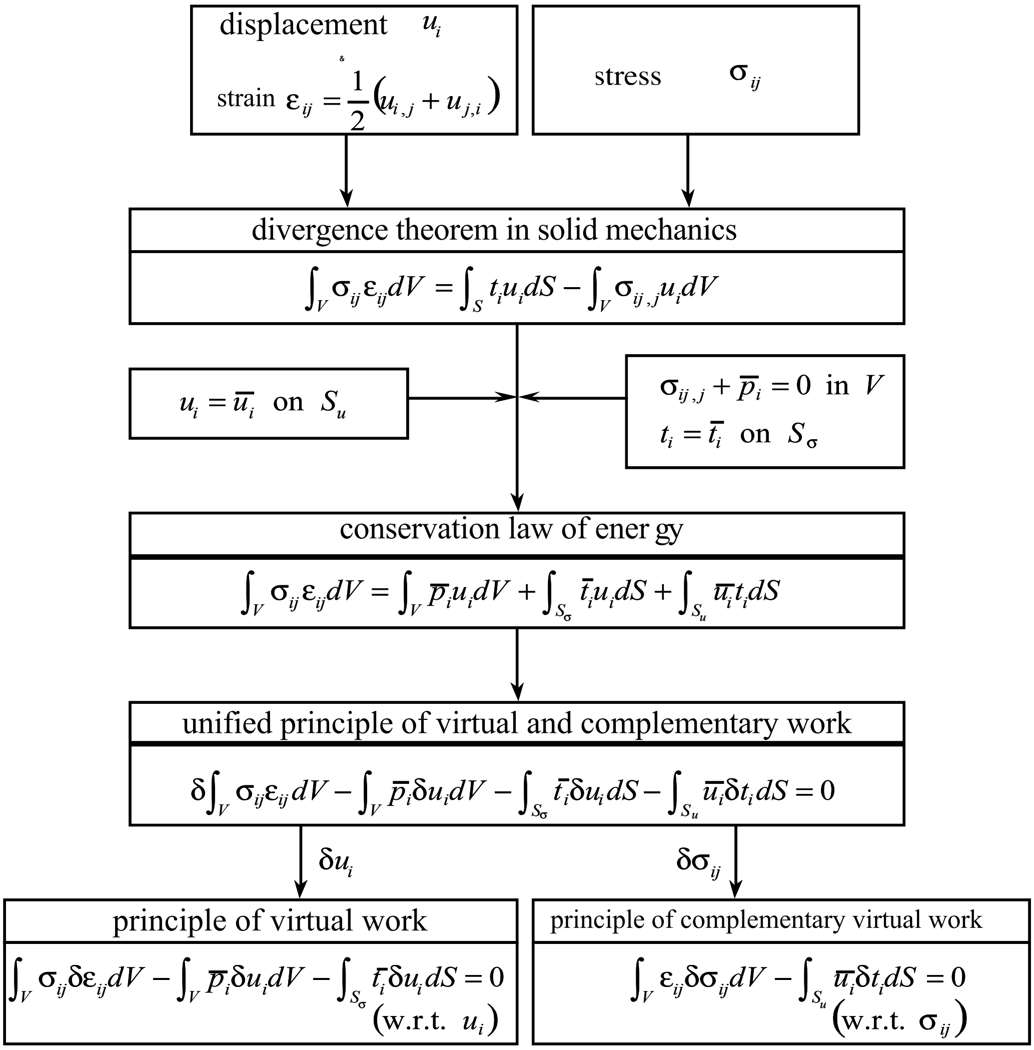
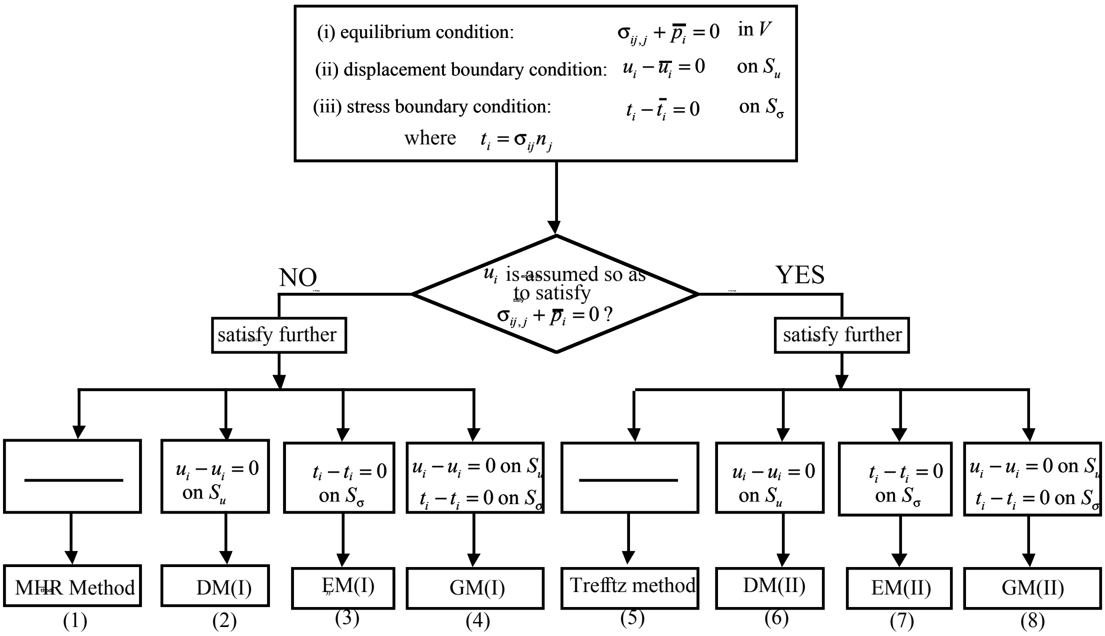


Fig.2 Process for proposing the unified principle of virtual work and complementary virtual work



remarks:

MHR Method: Modified Hellinger-Reissner Method

DM: Displacement Method

EM: Equilibrium Method or Force Method

GM: Galerkin Method

FEM is mainly based on DM(I), while Pian's Mixed Method covers DM(II) and EM(II), GM(II) is semi-analytical method of solution.

Fig. 3 8 possible methods of solution on solid mechanics problems

sol. No.	variational equations	constraint conditions	remarks
1	$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS+\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS - \int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0$	————	general method including other 7 methods
2	$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS-\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0$	$u_i-\bar{u}_i=0$ on S_u	DM(I)
3	$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS-\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0$	$t_i-\bar{t}_i=0$ on S_σ	EM(I)
4	$\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0$	$u_i-\bar{u}_i=0$ on S_u $t_i-\bar{t}_i=0$ on S_σ	GM (I)
5	$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS+\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS=0$	$\sigma_{ij,j}+\bar{p}_i=0$ in V	Trefftz's method
6	$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS=0$	$\sigma_{ij,j}+\bar{p}_i=0$ in V $u_i-\bar{u}_i=0$ on S_u	DM(II)
7	$\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS=0$	$\sigma_{ij,j}+\bar{p}_i=0$ in V $t_i-\bar{t}_i=0$ on S_σ	EM(II)
8	————————————	$\sigma_{ij,j}+\bar{p}_i=0$ in V $t_i-\bar{t}_i=0$ on S_σ $u_i-\bar{u}_i=0$ on S_u	GM(II) analytical solution

remarks:

DM: Displacement Method

EM: Equilibrium Method

GM: Galerkin Method

(I) u_i does not satisfy $\sigma_{ij,j}+\bar{p}_i=0$ a priori

(II) u_i satisfies $\sigma_{ij,j}+\bar{p}_i=0$ a priori

Table 1 8 possible methods of solution derived by the present variational formulation