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Development of a Nodeless and Consistent Finite Element Method
– force method forever –

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Abstract
Using the generalized potential which includes work done due of enforced displacement on the displacement boundary, a new variational formulation is attempted in this paper to unify the minimum principles of potential energy and complementary energy without introduction of Lagrange multiplier so that the minimal condition of the total energy can be asserted in true deformation of elastic bodies.
It is demonstrated solving a number of the plane stress and plate bending problems that

In any linear elasticity problem the lower bound solution can be always obtained by the method proposed in this paper with increase of NDOF of the test functions used.

To be paralleled with this research, a nodeless method is proposed in order to make the finite element calculation (especially force method) locking free and to reduce burden of the mesh generation problem in practice.

1 Introduction

Today, Force Method is almost declined in the FEM community. From the structural design point of view, however, the stiffness evaluation of structures based on the Displacement Method is not generally conservative and accuracy of the calculated stresses is inferior to that of the calculated displacements due to necessary differentiation of the latter. And therefore it is a long dream of structural engineers to establish a new method by which the lower bound solution can be obtained without fail in any elasticity problem. Toward such mission impossible, challenge has been made for the last 10 years conducting researches along the following two lines:

(i) development of a new variational formulation which unifies the minimum principles of potential and complementary energy so that the lower bound solution can be obtained at least in any linear elasticity problem.

(ii) development of a new nodeless method which makes the finite element analysis easier and more effective.

A few numerical examples on the plane stress and plate bending problems are introduced for verification of the proposed method in this paper.

2 Development of the unified energy method in elasticity

2.1 Generalization of the potential energy [3]

Consider arbitrary sets of stress components σ_{ij} and strain components ε_{ij}. The stress components σ_{ij} are assumed to satisfy the following equation of equilibrium Eq.(1) and the mechanical boundary condition on the stress boundary Eq.(2), while the strain ε_{ij} is assumed to be derived from the displacements using Eq.(3). The displacements u_i is also assumed to satisfy the displacement boundary conditions (4).

$$\sigma_{ij,j} + \overline{p}_i = 0 \qquad \text{in } V \tag{1}$$

where $\overline{p}_i$ is the body force vector, V is the volume of a given body.

$$t_i = \sigma_{ij} n_j = \bar{t}_i \qquad \text{in } S_\sigma \tag{2}$$

where n_i is the unit normal drawn outward on the stress boundary S_σ.

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \quad \text{in } V \tag{3}$$

$$u_i = \overline{u}_i \qquad \text{on } S_u \tag{4}$$

where $S = S_\sigma + S_u$; S_u is the displacement boundary.

Then it is not too difficult to derive the following equation using the divergence theorem:

$$\int_V \sigma_{ij} \varepsilon_{ij} dV = \int_V \overline{p}_i u_i dV + \int_{S_\sigma} \bar{t}_i u_i dS + \int_{S_u} \overline{u}_i t_i dS \tag{5}$$

It should be mentioned here that this equation is true irrespective of the stress-strain relation and size of u_i. If σ_{ij} and ε_{ij} are related by the following linear relation:

or $$\left.\begin{aligned}\sigma_{ij} &= a_{ijkl}\varepsilon_{kl} \\ \varepsilon_{kl} &= b_{ijkl}\sigma_{ij}\end{aligned}\right\} \tag{6}$$

where a_{ijkl} and b_{ijkl} are symmetric matrices.

Eq.(5) expresses the law of energy conservation. In this equation, the right hand side implies work done due to given body force, surface traction and enforced displacement on the displacement boundary. Therefore the concept of potential energy W can be generalized as follows:

$$W = \int_V \bar{p}_i u_i dV + \int_{S_\sigma} \bar{t}_i u_i dS + \int_{S_u} \bar{u}_i t_i dS \tag{7}$$

2.2 Proposal of a new unified minimum principle of total energy in elasticity

Consider the following total energy of an elastic system as defined by:

$$D_t(u_i) = U - W \tag{8}$$

where $$U = \int_V \sigma_{ij}\varepsilon_{ij} dV = V + V_c \tag{9-a}$$

$$\left.\begin{aligned} V &= \frac{1}{2}\int_V \sigma_{ij}\varepsilon_{ij} dV = \frac{1}{2}\int_V a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} dV \\ V_c &= \frac{1}{2}\int_V \varepsilon_{ij}\sigma_{ij} dV = \frac{1}{2}\int_V b_{ijkl}\sigma_{ij}\sigma_{kl} dV \end{aligned}\right\} \tag{9-b}$$

Now, Eq.(8) can be written as follows:

$$D_t(u_i) = D_p(u_i) + D_c(\sigma_{ij}) \tag{10-a}$$

where $$D_p(u_i) = \int_V \frac{1}{2}\sigma_{ij}\varepsilon_{ij} dV - \int_V \bar{p}_i u_i dV - \int_{S_\sigma} \bar{t}_i u_i dS \tag{10-b}$$

$$D_c(\sigma_{ij}) = \int_V \frac{1}{2}\varepsilon_{ij}\sigma_{ij} dV - \int_{S_\sigma} \bar{u}_i t_i dS \tag{10-c}$$

It should be mentioned here that $D_c(\sigma_{ij})$ is a function of u_i because σ_{ij} is a linear function of u_i via Eqs.(3) and (6). Therefore if u_i is the true solution, by the minimum principles of potential and complementary energy, the following conclusion can be drawn:

$$D_t(u_i) \rightarrow min \quad \text{w.r.t. } u_i \tag{11}$$

Conversely consider the case when $D_t(u_i)$ becomes minimum with respect to u_i. Since $D_t(u_i)$ is a sum of two positive functions of u_i, therefore, if at least any one of them does not become minimum, then $D_t(u_i)$ can not be minimum. Thus it can be concluded that a new principle proposed in this section unifies the minimum principles of potential and complementary energy.

Next, let's consider the strong form of $\delta D_t(u_i) = 0$. Now $D_t(u_i)$ is given as follows:

$$D_t(u_i) = \int_V \sigma_{ij}\varepsilon_{ij} dV - \int_V \bar{p}_i u_i dV - \int_{S_\sigma} \bar{t}_i u_i dS - \int_{S_u} \bar{u}_i t_i dS \tag{12}$$

The first variation of Eq.(12) with respect to u_i is given by

$$\int_V (\sigma_{ij}\delta\varepsilon_{ij} + \delta\sigma_{ij}\varepsilon_{ij}) dV - \int_V \bar{p}_i \delta u_i dV - \int_{S_\sigma} \bar{t}_i \delta u_i dS - \int_{S_u} \bar{u}_i \delta t_i dS = 0$$

This equation is transformed using divergence theorem as follows:

$$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS+\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS-\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV-\int_V \delta\sigma_{ij,j}u_i dV=0 \quad (13)$$

The last volume integral is unnecessary term which annihilates the variational calculation and therefore it must be deleted.

$$\therefore \int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS+\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS-\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0 \quad (14)$$

This conclusion was verified by a few numerical experiments made recently and it will be discussed in the later section again.

3 Correlation study of a new variational equation derived and other existing principles [3,4]

It should be noted here that the author derived previously the following variational equation generalizing the principle of virtual work with the use of of Lagrange multiplier:

$$\int_{S_\sigma}(t_i-\bar{t}_i)\delta u_i dS-\int_{S_u}(u_i-\bar{u}_i)\delta t_i dS-\int_V(\sigma_{ij,j}+\bar{p}_i)\delta u_i dV=0 \quad (15)$$

Eq.(15) is the strong form of the modified Hellinger-Reissner's variational eqation:

$\delta D_R(\sigma_{ij},u_i,\lambda_i)=0$ w.r.t. σ_{ij}, u_i and Lagrange multiplier λ_i

where

$$D_R(\sigma_{ij},u_i,\lambda_i)=\int_V\left(\sigma_{ij}\varepsilon_{ij}-\frac{1}{2}b_{ijkl}\sigma_{ij}\sigma_{kl}-\bar{p}_i u_i\right)dV-\int_{S_\sigma}\bar{t}_i u_i dS-\int_{S_u}\lambda_i(u_i-\bar{u}_i)dS \quad (16)$$

That is, Eq.(15) is equivalent to $D_R(u_i)$ when σ_{ij} and $\lambda_i=t_i$ are related to u_i by Eqs.(2) and (3).

It is interesting to note that difference of Eqs.(14) and (15) is only plus or minus sign of the second term of both equations.

It may be concluded that Eq.(14) can give always the lower bound, while Eq.(15) can give only stationary solutions although results of numerical experiment made so far are limited.

It should be also emphasized here that Eq.(11) is derived without introducing Lagrange multiplier, and therefore it can assert the minimum property of $D_t(u_i)$ while $D_R(\sigma_{ij},u_i)$ is only the stationary principle.

4 Eight possible methods of solution in elasticity

Using Eq.(16) it was discussed in the recent author's paper [1] that 8 different methods of solutions can be proposed as shown in the Tab.1. A few comments are made on the Tab.1 as follows:

(i) Solutions ① and ⑤ (Trefftz's method) are unique methods where continuity of the element state vector is not required a priori so that they can be treated independently.

(ii) For the rest methods continuity of the element state vector is required a priori so that they may be called the generalized finite element method. In the next section general approach to construct the nodeless method will be explained briefly.

(iii) Solution ⑦ are so called Equilibrium Method(II) (Force Method). Survival of EM(II) can be expected using this method.

(iv) Solution ⑧ is not a approximate method but analytical method. Indeed it must be called "computer-aided analytical solution". The present author believes future impact of this method to basic science and technology.

5 Development of the nodeless finite element method

In section 3, 6 methods of solution other than the solutions ① and ⑤ require continuity of element state vectors (displacements and boundary tractions) a priori. Continuity of the element state vector is considered as follows :

Firstly consider identity of the following two functions defined in the region $a \le x \le b$ as shown in Fig.1:

$$f(x) = g(x) \quad (a \le x \le b) \tag{a}$$

From this figure, it can be seen:

$$f(x_i) = g(x_i) \quad (i = 1,2,3,4) \tag{b}$$

Namely equality of two functions are true at only a number of discrete points. If these two functions are expressed in Maclaurin series, (b) is replaced by:

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n=b}^{\infty} b_n x^n \quad \text{irrespective of } x \ (a \le x \le b) \tag{c}$$

$$\therefore \quad a_n = b_n \tag{d}$$

In practice Maclaurin series must be approximated by a polynomial of a finite order. This idea can be successfully applied to formulation of the nodeless finite element method as follows:

For simplicity, the plane stress problem is considered: (Fig.2), L.M.N are midpoints of the sides $\overline{AB}$, $\overline{BC}$, $\overline{CA}$. Now the second order polynomials of *x, y* is assumed for the element displacement function as follows:

$$\left.\begin{aligned} u(x,y) &= u_0 - y\chi_0 + \varepsilon_{x0}x + \frac{\gamma_{xy0}}{2}y + a_4x^2 + a_5xy + a_6y^2 \\ v(x,y) &= v_0 + x\chi_0 + \frac{\gamma_{xy0}}{2}x + \varepsilon_{y0}y + b_4x^2 + b_5xy + b_6y^2 \end{aligned}\right\} \tag{17}$$

where (u_0, v_0, χ_0) is the rigid body displacement vector of the coordinate origin, $(\varepsilon_{x0}, \varepsilon_{y0}, \gamma_{xy0})$ is the constant strain components of the element and NDOF of this element is 12.

Using Eq.(17), the following set of equations can be derived.

(a) strain components $(\varepsilon_x, \varepsilon_y, \gamma_{xy})$

$$\left.\begin{aligned} &\varepsilon_x = \frac{\partial u}{\partial x} = \varepsilon_{x0} + 2a_4x + a_5y, \quad \varepsilon_y = \frac{\partial v}{\partial y} = \varepsilon_{y0} + b_3 + 2b_6y \\ &\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{xy0} + (a_5 + 2b_2)x + (2a_6 + b_5)y \end{aligned}\right\} \tag{18}$$

(b) stress components $(\sigma_x, \sigma_y, \tau_{xy})$

$$\left.\begin{aligned}
\sigma_x &= \frac{E}{(1-\nu^2)}(\varepsilon_x + \nu\varepsilon_y) = \frac{E}{(1-\nu^2)}\left[(\varepsilon_{x0} + \nu\varepsilon_{y0}) + (2a_4 + \nu b_5)x + (a_5 + 2\nu b_6)y\right] \\
\sigma_y &= \frac{E}{(1-\nu^2)}(\nu\varepsilon_x + \varepsilon_y) = \frac{E}{(1-\nu^2)}\left[(\nu\varepsilon_{x0} + \varepsilon_{y0}) + (2\nu a_4 + b_5)x + (\nu a_5 + 2b_6)y\right] \\
\tau_{xy} &= \frac{E}{2(1+\nu)}\gamma_{xy} = \frac{E}{2(1+\nu)}\left[\gamma_{xy0} + (a_5 + 2b_4)x + (2a_6 + b_5)y\right]
\end{aligned}\right\} \qquad (19)$$

(c) boundary traction components (t_x, t_y) $\quad t_x = \sigma_x l + \tau_{xy} m, \quad t_y = \tau_{xy} l + \sigma_y m$

$$\left.\begin{aligned}
t_x(x,y) &= \frac{E}{(1-\nu^2)}\left[\left\{l(\varepsilon_{x0} + \nu\varepsilon_{y0}) + \frac{(1-\nu)m}{2}\gamma_{xy0}\right\} + \left\{(2a_4 + \nu b_5)l + \frac{(1-\nu)m}{2}(a_5 + 2b_4)\right\}x \right.\\
&\qquad \left. + \left\{(a_5 + 2\nu b_6)l + \frac{(1-\nu)m}{2}(2a_6 + b_5)\right\}y\right] \\
t_y(x,y) &= \frac{E}{(1-\nu^2)}\left[\left\{\frac{(1-\nu)l}{2}\gamma_{xy0} + (\nu\varepsilon_{x0} + \varepsilon_{y0})m\right\} + \left\{\frac{(1-\nu)l}{2}(a_5 + 2b_4) + m(2\nu a_4 + b_5)\right\}x \right.\\
&\qquad \left. + \left\{\frac{(1-\nu)l}{2}(2a_6 + b_5) + (\nu a_5 + 2b_6)m\right\}y\right]
\end{aligned}\right\} \qquad (20)$$

where $\boldsymbol{n}(l,m)$ is an unit normal drawn outward on the boundaries of a given triangular element as shown in Fig.2.

Denoting the midpoint of $\overline{AB}$ by $L(x_L, y_L)$, an arbitrary point $P(x,y)$ is taken on $\overline{AB}$ as shown in Fig.2.

Putting the distance $\left|\overrightarrow{PL}\right| = s$, the following equation can be easily obtained:

$$\left.\begin{aligned}
x &= x_L - ms \\
y &= y_L + ls
\end{aligned}\right\} \qquad (21)$$

Eq.(21) is the equation for coordinate transformation of (x,y) into s on the side $\overline{AB}$. Applying Eq.(21) to Eqs.(17) and (20), the state vector on the side $\overline{AB}$ are given by:

$$\left.\begin{aligned}
u(s) &= A_0 + A_1 s + A_2 s^2 \\
v(s) &= B_0 + B_1 s + B_2 s^2
\end{aligned}\right\} \qquad (22)$$

$$\left.\begin{aligned}
t_x(s) &= C_0 + C_1 s \\
t_y(s) &= D_0 + D_1 s
\end{aligned}\right\} \qquad (23)$$

These constants A_n, B_n, C_n, D_n can be expressed by the following set of equations:

$$\left.\begin{aligned}
A_0 &= u(0) = a_0 + a_1 x_L + a_3 y_L + a_4 x_L + a_5 x_L y_L + a_6 y_L \\
A_1 &= u'(0) = \left| a_0\left(\frac{dx}{ds}\right) + a_3\left(\frac{dy}{ds}\right) + 2a_4\left(x\frac{dx}{ds}\right) + a_5\left(y\frac{dx}{ds}\right) + a_5\left(x\frac{dy}{ds}\right) + 2a_6\left(y\frac{dy}{ds}\right)\right|_{s=0} \\
&= -ma_2 + la_3 - 2mx_L a_4 + a_5(lx_L - my_L) + 2y_L a_6 \\
A_2 &= \frac{1}{2}u''(0) = \left|\frac{1}{2}\left\{2a_4\left(\frac{dx}{ds}\right)^2 + 2a_5\left(\frac{dx}{ds}\right)\left(\frac{dy}{ds}\right) + 2a_6\left(\frac{dy}{ds}\right)^2\right\}\right|_{s=0} = ma_4 - lma_5 + l^2 a_5
\end{aligned}\right\} \qquad (24)$$

similarly

$$\left.\begin{aligned} B_0 &= b_1 + b_2 x_L + b_3 y_L + b_4 {x_L}^2 + b_5 x_L y_L + b_6 {y_L}^2 \\ B_1 &= -m b_2 + l b_3 - 2 m x_L b_4 + b_5 (l x_L - m y_L) + 2 l y_L b_5 \\ B_2 &= m^2 b_4 - l m b_5 + l^2 b_6 \end{aligned}\right\} \tag{25}$$

$$\left.\begin{aligned} & C_0 = P + Q x_L + R y_L \quad C_1 = -mQ + Rl \\ & \textit{where} \quad P = \frac{E}{(1-\nu^2)}\left\{ l(\varepsilon_{x0} + \nu \varepsilon_{y0}) + \frac{(1-\nu)m}{2}\gamma_{xy0} \right\} \\ & Q = \frac{E}{(1-\nu^2)}\left\{ \frac{(1-\nu)l}{2}(2a_4 + \nu b_5)l + \frac{(1-\nu)m}{2}(a_5 + 2b_4) \right\} \\ & R = \frac{E}{(1-\nu^2)}\left\{ (a_5 + 2\nu b_6)l + \frac{(1-\nu)m}{2}(2a_6 + b_5) \right\} \end{aligned}\right\} \tag{26}$$

$$\left.\begin{aligned} & D_0 = S + S x_L + U y_L , \quad D_1 = -mT + Ul \\ & \textit{where} \quad S = \frac{E}{(1-\nu^2)}\left\{ \frac{(1-\nu)l}{2}\gamma_{xy0} + (\nu \varepsilon_{x0} + \varepsilon_{y0})m \right\} \\ & T = \frac{E}{(1-\nu^2)}\left\{ \frac{(1-\nu)l}{2}(a_5 + 2b_4) + (2\nu a_4 + b_5)m \right\} \\ & U = \frac{E}{(1-\nu^2)}\left\{ \frac{(1-\nu)l}{2}(2a_6 + b_5) + (\nu a_5 + 2b_6)m \right\} \end{aligned}\right\} \tag{27}$$

Since NDOF of this triangular element is 12 and only 4 components can be distributed to each side. Therefore the following three cases may be feasible for this triangular element.

(i) Displacement Model (I) (DM I) $\quad u(s) = A_0 + A_1 s, \quad v(s) = B_0 + B_1 s$

(ii) Mixed Model (I) (MM I) $\quad u(s) = A_0, \quad v(s) = B_0, \quad t_x(s) = C_0, \quad t_y(s) = D_0$

(iii) Equilibrium Model (I) (EM I) $\quad t_x(s) = C_0 + C_1 s, \quad t_y(s) = D_0 + D_1 s$

where (I) implies the model which does not satisfy the equation of equilibrium, while models composed basing on the stress functions ϕ_i are referred to (II).

Now in case of DM I , Eqs.(24) and (25) can be set up for each side of the element ΔABC. Therefore 12 element parameters $(u_0, v_0, \chi_0; \varepsilon_{x0}, \varepsilon_{y0}, \gamma_{xy0}, a_1, a_2, a_3, b_1, b_2, b_3)$ can be transformed into 3 sets of (A_0, A_1, B_0, B_1) by a transformation matrix $\boldsymbol{A}(12 \times 12)$. And therefore if the element stiffness matrix $\overline{\boldsymbol{K}}$ is obtained with respect to the local coordinate, the stiffness matrix $\boldsymbol{K}$ with respect to new element parameters $({A_0}^{(k)}, {A_1}^{(k)}, {B_0}^{(k)}, {B_1}^{(k)})$ $(k = 1,2,3)$ can be automatically obtained by the following familiar formula:

$$\boldsymbol{K} = \boldsymbol{A}^T \overline{\boldsymbol{K}} \boldsymbol{A} \tag{28}$$

where the superscript $k = 1,2,3$ implies sides of ΔABC. $\overline{AB}, \overline{BC}$ and $\overline{CA}$ respectively.
Similarly the element flexibility matrix can be obtained.
Generally speaking models belong to category (MM I) may give results of the better convergency and accuracy.

Practical development of nodeless finite elements for 2D as well as 3D analyses will be systematically made. For example, consider the plate bending problem. The state vector consists of 4 components.

$$\left(w, \frac{\partial w}{\partial n}, M_n \ and \ V_n \right)$$

In conventional finite element method the compatible model can hardly give accurate results. In case of the present method, however, no reason can be seen which may produce inferior results to compare with the existing finite element method. In case of the plate bending problem continuity conditions of the state vectors consists of four equations. The brief introduction of the plate bending analysis will be given as follows :

(1) The equilibrium equation and boundary conditions (see Fig.3)

Equilibrium equation: $D\Delta\Delta w(x,y) = \overline{q}(x,y)$

Associated boundary conditions:

displacement b.c.: $w = \overline{w}, \quad \frac{\partial w}{\partial n} = \frac{\partial \overline{w}}{\partial n} \qquad on\ C_w$

stress b.c.: $M_n = \overline{M}_n, \ V_n = \overline{V}_n \qquad on\ C_m$

where n is the unit normal drawn outward to the boundary $\boldsymbol{n} = (l, m)$, $C = C_w + C_m$

C_w : displacement b.c. C_m : stress b.c.

$C_w = C_{w0} + C_{w1}, \quad C_m = C_{m0} + C_{m1}$

C_{w0} : w prescribed b.c. C_{w1} : $\frac{\partial w}{\partial n}$ prescribed b.c.

C_{m0} : V_n prescribed b.c. C_{m1} : M_n prescribed b.c.

$$M_n = M_x l^2 + M_y m^2 + 2M_{xy} lm$$

$$M_{nt} = M_{xy}\left(l^2 - m^2\right) + \left(M_x - M_y\right)m$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\nabla^2 w\right), \ Q_y = -D\frac{\partial}{\partial y}\left(\nabla^2 w\right)$$

$$Q_n = Q_x l + Q_y m, \quad V_n = Q_n + \frac{\partial M_{nt}}{\partial s}$$

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \ M_y = -D\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = -D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}$$

(2) Minimum principle of the total energy for plate bending problems

$\ddot{a}D_t(w) = 0, \quad \ddot{a}^2 D_t(w) \geq 0 \quad w.r.t\ w$

where

$$D_t(w) = -\iint_D \left(M_x \frac{\partial^2 w}{\partial x^2} + M_y \frac{\partial^2 w}{\partial y^2} + 2M_{xy}\frac{\partial^2 w}{\partial x \partial y}\right)dxdy - \iint_D \overline{q} w dxdy$$
$$+ \int_{C_{m1}} \overline{M}_n \frac{\partial w}{\partial n} ds - \int_{C_{m0}} \overline{V}_n w ds + \int_{C_{m1}} M_n \frac{\partial \overline{w}}{\partial n} ds - \int_{C_{m0}} V_n \overline{w} ds \qquad (29)$$

This equation can be transformed into the following equation:

$$D_t(w) = \iint_D D\left[\left(\nabla^2 w\right)^2 - 2(1-\nu)\left\{\left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right\}\right]dxdy$$

$$-\iint_D \overline{q} w dxdy + \int_{C_{m1}} \overline{M}_n \frac{\partial w}{\partial n} ds - \int_{C_{m0}} \overline{V}_n w ds + \int_{C_{m1}} M_n \frac{\partial \overline{w}}{\partial n} ds - \int_{C_{m0}} V_n \overline{w} ds \tag{30}$$

Using the divergence theorem, Eq.(29) can be transformed into the following strong form:

$$D_t(w) = \iint_D \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2M_{xy}\frac{\partial^2 M_{xy}}{\partial x \partial y} + \overline{q}\right) w dxdy - \int_{C_{m1}} \left(M_n - \overline{M}_n\right)\frac{\partial w}{\partial n} ds$$

$$+\int_{C_{m0}} \left(V_n - \overline{V}_n\right) w ds - \int_{C_{m1}} M_n \left(\frac{\partial w}{\partial n} - \frac{\partial \overline{w}}{\partial n}\right) ds + \int_{C_{m0}} V_n \left(w - \overline{w}\right) ds \tag{31}$$

Therefore

$$\delta D_t(w) = \iint_D \left(\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2M_{xy}\frac{\partial^2 M_{xy}}{\partial x \partial y} + \overline{q}\right) \delta w dxdy$$

$$-\int_{C_{m1}} \left(M_n - \overline{M}_n\right)\delta\left(\frac{\partial w}{\partial n}\right) ds + \int_{C\sigma 0} \left(V_n - \overline{V}_n\right)\delta w ds \quad \text{w. r. t. w} \tag{32}$$

$$-\int_{C_{w1}} \delta M \left(\frac{\partial w}{\partial n} - \frac{\partial \overline{w}}{\partial n}\right) ds + \int_{C_{w0}} \delta V_n \left(w - \overline{w}\right) ds = 0$$

The general solution of $D\Delta\Delta w(x,y) = 0$ can be obtained following the duality law between the equations of the plane stress and plate bending problems as follows:

$$\left.\begin{aligned}
&w(x,y) = \mathrm{Re}\left[\overline{z}\varphi(z) + \chi(z)\right] \\
&D\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right) = z\varphi'(z) + \chi'(z) + \overline{\varphi}(\overline{z}) \\
&M_x = -2(1+\nu)\mathrm{Re}\left[\varphi'(z)\right] - (1-\nu)\mathrm{Re}\left[\overline{z}\varphi''(z) + \chi''(z)\right] \\
&M_y = -2(1+\nu)\mathrm{Re}\left[\varphi'(z)\right] + (1-\nu)\mathrm{Re}\left[\overline{z}\varphi''(z) + \chi''(z)\right] \\
&M_{xy} = (1-\nu)\mathrm{Im}\left[\overline{z}\varphi''(z) + \chi''(z)\right] \\
&Q_x + iQ_y = -4\varphi'(z)
\end{aligned}\right\} \tag{33}$$

Now Eq.(32) is the variational equation for the plate bending problem without any associated boundary condition and it is derived without using Lagrange multiplier method. Again 8 possible methods of solution can be proposed and exactly the same discussions can be applied to this problem as made in the plane stress analysis:

Methods of solution ① and ⑤ present unique solution procedure for analysis of a discrete system where the element state vectors can be assumed independently.

In the other cases continuity of element state vector four components ($w, \frac{\partial w}{\partial n}, M_n, V_n$) are partially or fully required on the element boundaries before analysis of the total systems. (generalized finite element method).

In the conventional finite element method it is extremely difficult to satisfy even the displacement continuities of w and $\frac{\partial w}{\partial n}$ simultaneously on the element boundaries. It should be again emphasized that the characteristic element matrices (for the mixed model) of any shape and order can be constructed systematically in the present method.

Consider, for example, the consistent quadrilateral plate bending element. At least it must satisfy ontinuity of w (3rd order) , $\frac{\partial w}{\partial n}$ (2nd order), M_n (1st order) and V_n (0 order) on each side of the element, therefore total NDOF of the element is $4\times(1+2+3+4)=40$. Consequently the 11th order polynomial of x and y must be assumed for $w(x,y)$.

6 Numerical Examples

About 7 years ago the author has initiated a ambitious challenge to establish a new variational formulation by which the lower bound solution can be always obtained in any elasticity problem via generalization of the principle of virtual work.
A few years later, he recognized that his new variational formulation is essentially equivalent to Hellinger-Reissner's Principle $D_{HR}(u_i, \sigma_{ij}, \varepsilon_i)$ if σ_{ij} and λ_i are eliminated using the stress-strain law and the associated extremum condition $\lambda_i = t_i = \sigma_{ij} n_j$, and it is only stationary principle with respect to u_i. Basing on such consideration he concluded that in order to restore the force method further challenge must be continued to search a new variational principle which may unify the minimum principles of the potential and complementary energy without introducing Lagrange multiplier.
In what follows brief introduction will be given to the five numerical examples of verification studies.

Example (I) Analysis of the stress concentration problem of a perforated square plate subjected to uniaxial tension at the ends [4]

This problem is analyzed using Trefftz's method based on the generalized principle of virtual work. One element solution as well as finite element solution are shown in the Fig.4.
Distribution of σ_x on the $\overline{CC'}$ cross section is shown in this figure. Accuracy and good agreement of calculated results between one element and finite element solutions are observed. It should be especially mentioned that free combination of mesh pattern with different size and shape can be easily done in the present method because it is a nodeless method based on Trefftz's solution procedure.

Example (II) Torsion analysis of an elastic bar with a square cross section using the membrane analogy

Torsion problem of an elastic bar with the cross section of arbitrary shape can be given by the following boundary value problem of Poisson equation with Derichlet boundary condition:

$$\left.\begin{aligned} &\nabla^2\phi(x,y) = -2G\theta \quad && in\ S \\ &\phi = 0 && on\ C \end{aligned}\right\} \tag{34}$$

where $\phi(x,y)$ is the stress function for torsion, θ is the rate of twist, and the twisting moment M_t is given by:

$$M_t = 2\iint \phi\, dxdy$$

Due to the membrane analogy proposed by L.Prandtl, Eq.(34) is equivalent to the following deflection problem of the elastic membrane under uniform lateral load $\overline{q}$.

$$\left.\begin{array}{ll}\nabla^2 w(x,y) = -\dfrac{\overline{q}}{T} & \\ w = 0 & on\ C\end{array}\right\}$$

where T is the tension in the membrane.

Now the associated boundary condition is given by:

$w = \overline{w} \quad on\ C$ (Derichlet condition)

The functional $D_t(w)$ for this problem can be given by:

$$D_t(w) = \iint_D T\left\{\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right\} dxdy - \iint_D \overline{q}\, dxdy - \int_C \overline{w}\, t_n ds \tag{35}$$

where $t_n = T\dfrac{\partial w}{\partial n}$

The strong form of $\delta D_t(w) = 0$ is given by:

$$\int_C (w - \overline{w})\delta t_n ds - \iint_D \left(T\nabla^2 w + \overline{q}\right)\delta w dxdy = 0 \tag{36}$$

It is customary to assume $\overline{w} = 0$. Therefore the problem is reduced to a simple exercise problem for Rayleigh-Ritz's method. Torsion of an elastic bar with a square cross section was analyzed using the following deflection function of the 3rd order polynomials of x and y :

$$w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 \tag{37}$$

Two results of calculation for the torsional rigidity K are shown in Fig.5. The curve –■– is the result of calculation using the second order polynomials of x and y, while the result obtained using the 3rd polynomials are shown by the curve –◆–.

It is interesting to note that the former analysis gave the upper bound solution, while the latter gave the lower bound solution for K. It can be seen that both calculations shew monotonous convergency to the exact $K = 0.1406$ (Timoshenko) assuming the side length $2a = 1$.

Example (III) Inplane bending analysis of a cantilever plate subjected to a boundary shear [2,4]

Timoshenko studied bending of a cantilever having narrow rectangular cross section of unit width bent by a force applied at the end (Fig.6). To solve this problems the effect of a force P is approximated by a distributed shearing stress acting at the end as follows:

$$\left.\begin{array}{l}\sigma_x(l,y) = 0 \\ \tau_{xy}(l,y) = -\dfrac{P}{2I}\left(c^2 - y^2\right)\end{array}\right\} \tag{38}$$

This problem was analyzed recently using the unified energy method proposed in this paper. Finite element analyses were conducted using the following nonequilibrium displacement function, NDOF of which is 16 as follows:

$$\left.\begin{array}{l}u(x,y) = u_0 - \chi_0 y + \varepsilon_{x0} x + \dfrac{1}{2}\gamma_{xy0} y + a_1 x^2 + a_2 xy + a_3 y^2 + a_4 x^2 y + a_5 xy^2 \\ v(x,y) = v_0 + \chi_0 x + \varepsilon_{y0} y + \dfrac{1}{2}\gamma_{xy0} x + b_1 x^2 + b_2 xy + b_3 y^2 + b_4 x^2 y + b_5 xy^2\end{array}\right\} \tag{39}$$

For the equilibrium displacement functions of the same NDOF is also derived using 4th order polynomial of z for $\varphi(z)$ and $\chi(z)$ of the following Goursat's stress function. In brief

Airy's stress function $F(x,y)=\mathrm{Re}[\bar{z}\varphi(z)+\chi(z)]$

$$\varphi(z)=\sum_{n}^{\infty}A_n z^n,\quad \chi(z)=\sum_{n}^{\infty}B_n z^n$$

where
$$\left.\begin{aligned}
&z=x+iy,\quad A_n=a_n+ib_n,\quad B_n=c_n+id_n\\
&\sigma_x+\sigma_y=4\,\mathrm{Re}[\varphi'(z)]\\
&\sigma_y-\sigma_x+2i\tau_{xy}=2[\bar{z}\varphi''(z)+\chi''(z)]\\
&2G(u+iv)=\left(\frac{3-\nu}{1+\nu}\right)\varphi(z)-z\varphi'(z)-\chi'(z)
\end{aligned}\right\}\tag{40}$$

A solution obtained using nonequilibrium displacement function is shown by the curve –■– , while the other solution using the equilibrium displacement is shown by the curve –◆– in this figure. Fig.6 show the convergency of the calculated displacement v_A and stress σ_B respectively. It can be seen that the curves –■– gives always the upper bound solution for both v_A and σ_B, on the other hand, the curve –◆–gives the lower bound solution clearly.
The robustness of the present method was duly checked numerically. It should be mentioned that analytical 2D solution given by Timoshenko is not exact solution for the problem but elaborate approximate solution where the clamped edge condition is approximated by clamping the plate at the origin. (Indeed substantial difference of deflection v_A is observed in Fig.6)

Example (IV) Bending of a square plate subjected to uniformly distributed lateral loading with all four edges clamped [4]

This problem is historically well-known difficult problem to which many scholars have attacked in the past. Among them Timoshenko work published in 1938 is the most well known. This problem was analyzed using the new variational method proposed in this paper. The 10th order polynomial of *x, y* is used for the element deflection function. Results obtained are summarized in Fig.7.

Example (V) Bending of a square plates subjected to uniformly distributed lateral loading with all four edges simply supported and its 1st eigenvalue analysis of the bending vibration [4]

Fig.8 shows size, mesh divisions and calculated deflection at the center. Displacement functions used are polynomials of 5th order where two axes symmetry of the plate deflection is considered. The abscissa of the convergency curves are total number of degree of freedom (NDOF of the element used ×number of elements). It can be clearly seen that solutions obtained are always the lower bound solutions in the plate bending analysis.
The eigenvalue equation for the plate bending vibration is given by:

$$\Delta\Delta w=\lambda^4 w$$

substituting $\lambda^4 w$ for $\bar{q}$ in $D_t(w)$ for the plate bending problem and making minimum with respect to unknown parameters a_n of $w(x,y)$, homogeneous linear equations for a_n can be obtained, from which the characteristic equation for λ can be obtained eliminating a_n. 1st eigenvalues obtained (lower bound solutions) are shown in the table attached to Fig.9.

7 Conclusions

Using divergence theorem in elasticity, a new variational principle was proposed on the minimum condition of the total energy of a given elastic system.
This method may unify the minimum principles of potential and complementary energy in the linear elasticity so that accuracy of approximate solutions can be definitely checked bracketing them by the upper bound and lower bound solutions.
It is interesting note that difference between the proposed principle and the well established Hellinger-Reissner's Principle is only difference of the sign of work done due to enforced displacement on the displacement boundaries and yet the former can assert the minimum of the total energy, while the latter is only a stationary principle. However, rigorous mathematical proof is left for future study.

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Table1: 8 Possible methods of solution derived from the present variational formulation

SOL. NO.	Variation equations	Constraint Conditions	remarks
1	$\int_V (\sigma_{ij,j} + \overline{p}_i)\delta u_i dV - \int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS - \int_{S_u} (u_i - \overline{u}_i)\delta t_i dS = 0$	—	Pair of modified Hellinger-Reissner's method
2	$\int_V (\sigma_{ij,j} + \overline{p}_i)\delta u_i dV - \int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS = 0$	$u_i - \overline{u}_i = 0$ on S_u	Displacement Method (I) (DM I)
3	$\int_V (\sigma_{ij,j} + \overline{p}_i)\delta u_i dV - \int_{S_u} (u_i - \overline{u}_i)\delta t_i dS = 0$	$t_i - \bar{t}_i = 0$ on S_σ	Equilibrium Method (I) (EM I)
4	$\int_V (\sigma_{ij,j} + \overline{p}_i)\delta u_i dV = 0$	$u_i - \overline{u}_i = 0$ on S_u $t_i - \bar{t}_i = 0$ on S_σ	Galerkin's Method (I)
5	$\int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS + \int_{S_u} (u_i - \overline{u}_i)\delta t_i dS = 0$	$\sigma_{ij,j} + \overline{p}_i$ in V	Trefftz's Method
6	$\int_{S_\sigma} (t_i - \bar{t}_i)\delta u_i dS = 0$	$\sigma_{ij,j} + \overline{p}_i$ in V $u_i - \overline{u}_i = 0$ on S_u	DM(II)
7	$\int_{S_u} (u_i - \overline{u}_i)\delta t_i dS = 0$	$\sigma_{ij,j} + \overline{p}_i$ in V $t_i - \bar{t}_i = 0$ on S_σ	EM(II)
8	—	$\sigma_{ij,j} + \overline{p}_i$ in V $t_i - \bar{t}_i = 0$ on S_σ $u_i - \overline{u}_i = 0$ on S_u	GM(II) analytical solution

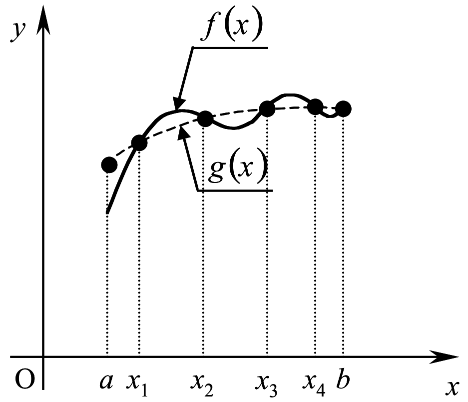


Figure 1: Equality of two functions: $f(x) = g(x)$ from the collocation method point of view

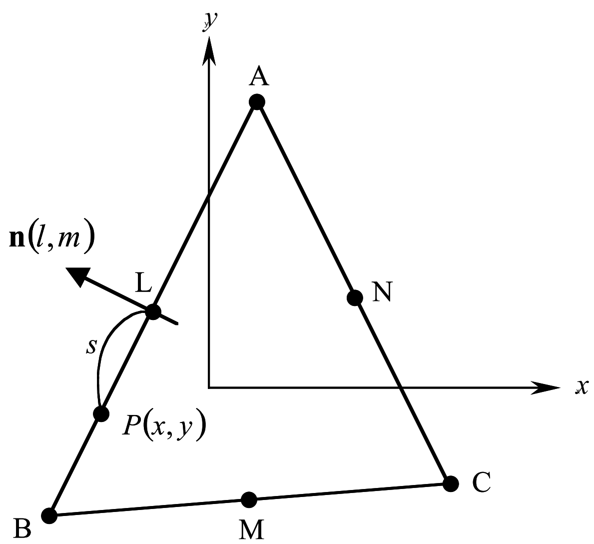


Figure 2: Development of a nodeless finite element

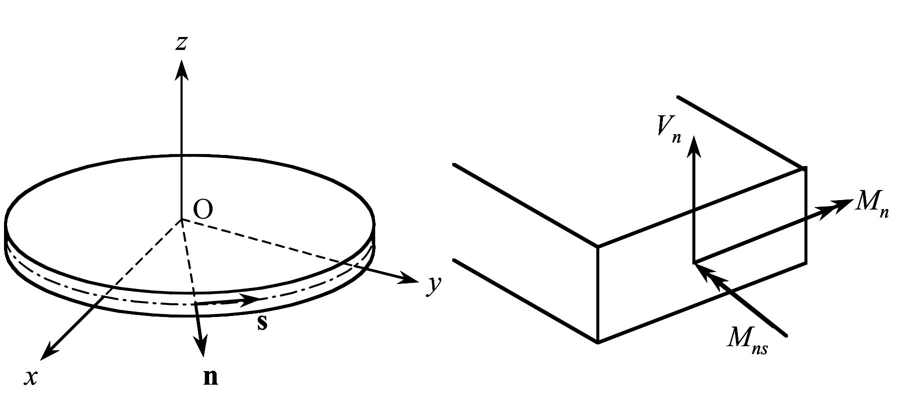
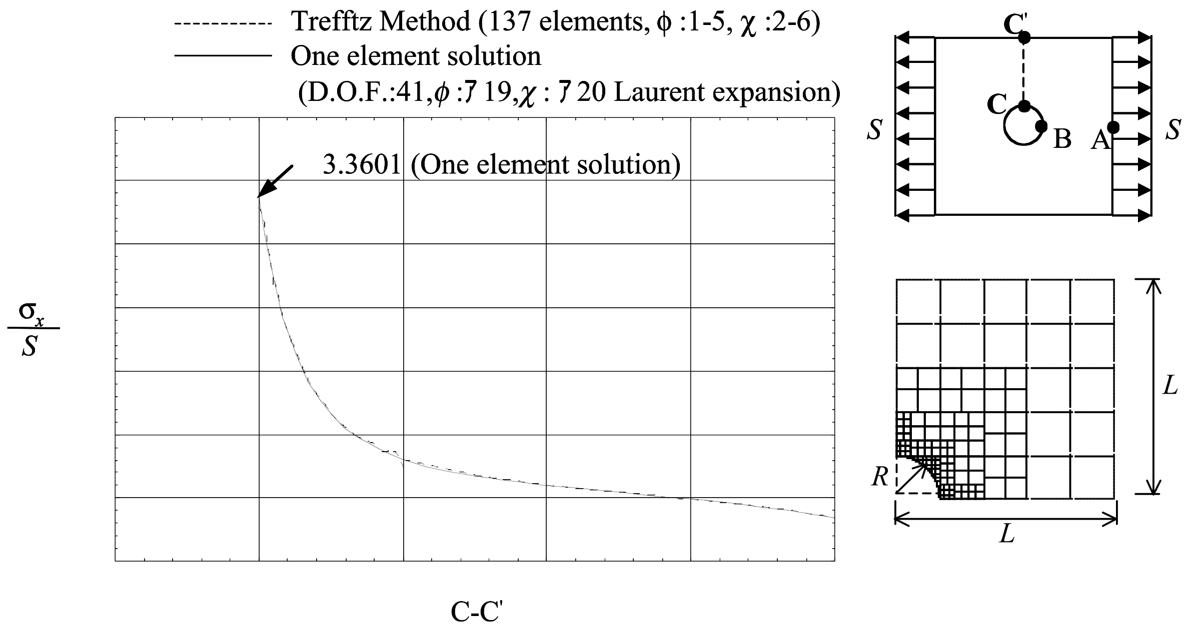


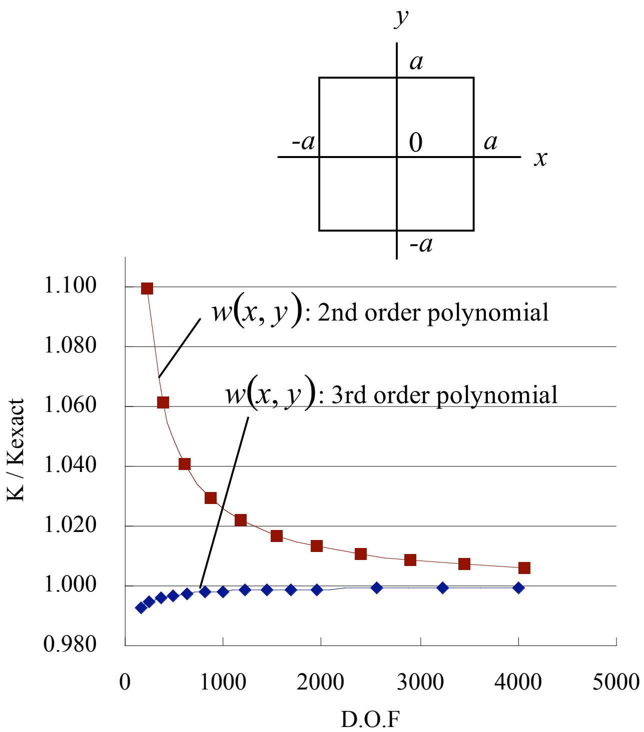
Figure 3: Coordinates and components of the traction vector for the plate bending problem



$R = 10\text{mm}, \quad L = 50\text{mm}, \quad S = 100\text{kgf/mm},$

$E = 20000\text{kgf/mm}^2, \quad \nu = 0.3$

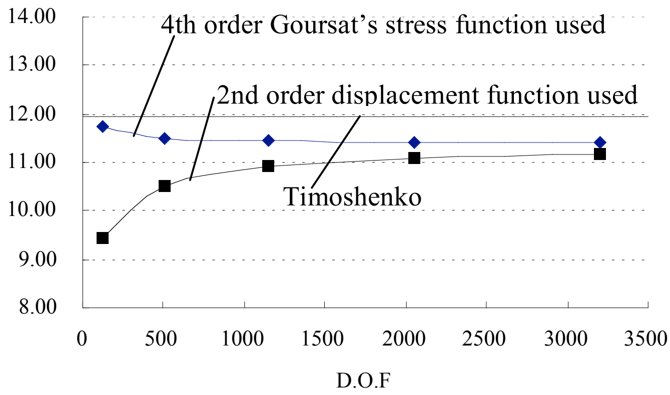
Figure 4: Stress distribution on section C-C' of a perforated square plate under uniaxial uniform loading

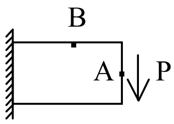


6DOF / element		10 DOF / element	
Mesh Div.	2nd order polynomial	Mesh Div.	3rd order polynomial
2×2	0.17708	3×3	0.13951
4×4	0.16498	4×4	0.13961
6×6	0.15456	5×5	0.13984
8×8	0.14921	6×6	0.14002
10×10	0.14635	7×7	0.14014
12×12	0.14469	8×8	0.14023
14×14	0.14365	9×9	0.14030
16×16	0.14295	10×10	0.14035
18×18	0.14247	11×11	0.14039
20×20	0.14211	12×12	0.14041
22×22	0.14185	13×13	0.14044
24×24	0.14165	14×14	0.14046
26×26	0.14149	16×16	0.14048
28×28	0.14137	18×18	0.14050
30×30	0.14127	20×20	0.14052
Timoshenko K=0.1406(2a)^4			

Figure 5: Analysis of torsional rigidity of an elastic bar with the square cross section (divided by square mesh)

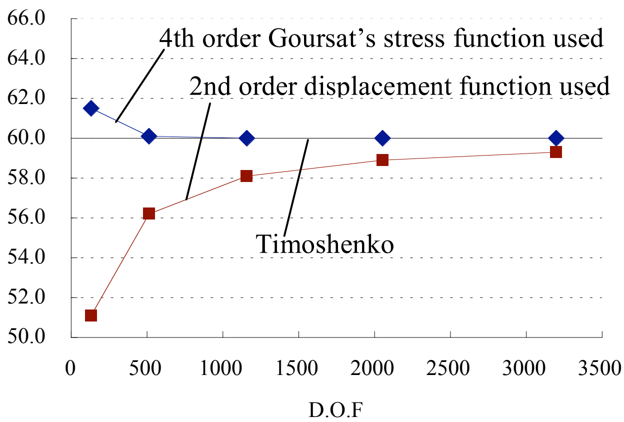
v_A : vertical displacement at the point A

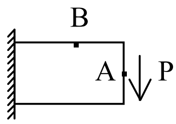




Mesh Div. × NDOF	stress function used	displacement function used
4×2×16	11.7195	9.4399
8×4×16	11.4996	10.5163
12×6×16	11.4347	10.9196
16×8×16	11.4063	11.0912
20×10×16	11.3909	11.1780

$(\sigma_x)_B$: stress at the point B





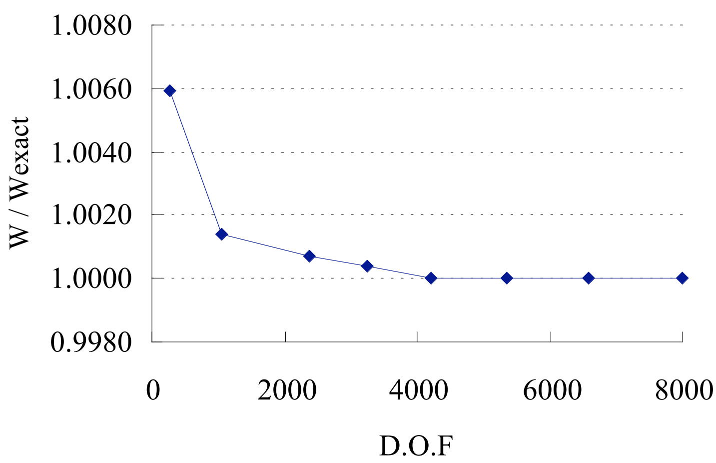
Mesh Div. × NDOF	stress function used	displacement function used
4×2×16	61.4766	51.0777
8×4×16	60.0641	56.1607
12×6×16	60.0287	58.1254
16×8×16	60.0138	58.8946
20×10×16	60.0071	59.2698

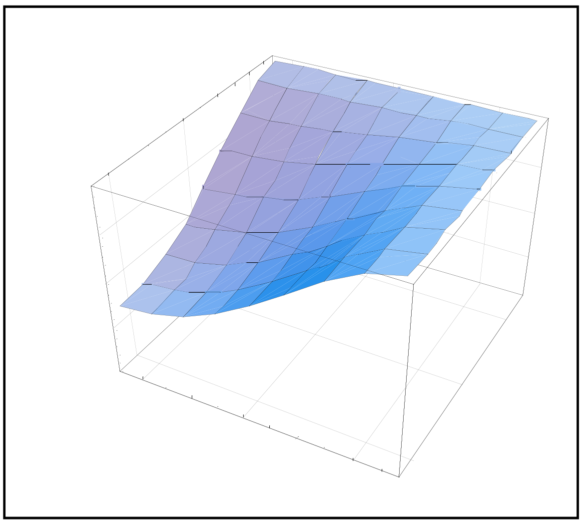
Figure 6: Inplane bending analysis of a cantilever plate subjected to a boundary shear of parabolic distribution (divided by square mesh)

$w(0,0)_{exact}$

Central deflection $w(0,0)$

Mesh Div ×NDOF		
2×2×66	1.0059	1.27247
3×3×66	1.0014	1.26676
4×4×66	1.0014	1.26675
5×5×66	1.0006	1.26576
6×6×66	1.0007	1.26587
7×7×66	1.0004	1.26555
8×8×66	1.0000	1.26524
9×9×66	1.0000	1.26502
10×10×66	1.0000	1.26531
11×11×66	1.0000	1.26531





deflection (Mesh Div. $\times$NDOF = $8\times8\times66$)

Figure 7: Finite element bending analysis of a square plate under uniformly distributed load using the newly proposed variational method.

Nonequilibrium 10th order polynomials of (x,y) were used for analysis.

Mesh Div. ×NDOF	Total D.O.F	deflection w(0,0)
2×2×21	84	4.5215
3×3×21	189	4.2231
4×4×21	336	4.1613
5×5×21	525	4.1336
6×6×21	756	4.1149
7×7×21	1029	4.1037
8×8×21	1344	4.0960
9×9×21	1701	4.0908
Timoshenko	4.061	

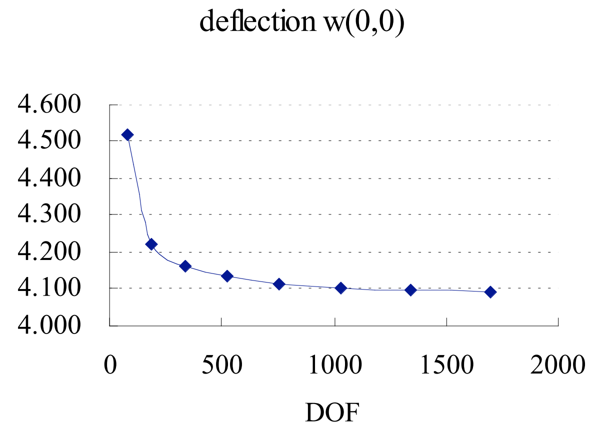


Figure 8: Bending analysis of a simply supported square plate displacement function used: 5th order polynomials of (x, y) (NDOF=21)

Mesh Div. ×NDOF	Total D.O.F	eigenvalue λ
2×2×21	84	121.73
3×3×21	189	121.94
4×4×21	336	122.09
5×5×21	525	122.17
6×6×21	756	122.21
7×7×21	1029	122.25
8×8×21	1344	122.27
9×9×21	1701	122.29
10×10×21	2100	122.30
11×11×21	2541	122.31
12×12×21	3029	122.32
13×13×21	3549	122.32
Exact	122.36	

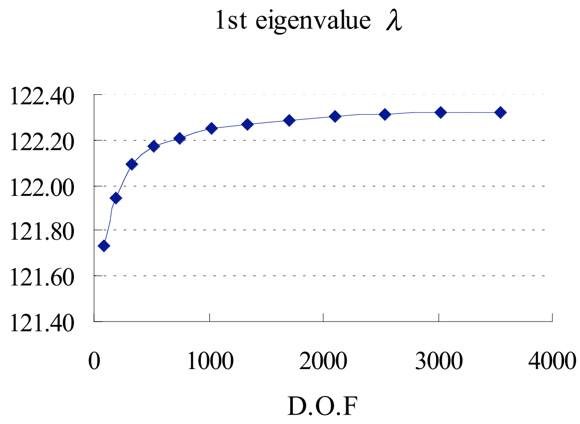


Figure 9: 1st eigenvalue analysis of a simply supported square plate displacement function used: 5th order polynomials of (x, y) (NDOF=21)